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A BANACH ALGEBRA APPROACH TO AMALGAMATED R- AND S-TRANSFORMS

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Abstract. We give a Banach algebra approach to the additiveness property of Voiculescu's amalgamated R-transform. We also define an amalgamated S-transform and prove that it is multiplicative on products of amalgamated free random variables when the algebra of amalgamation is commutative.

1. Introduction

If A is a unital algebra and ϕ is a unital linear functional on A we will say that (A, ϕ) is a free probability space. The elements of A are then the free random variables, and ϕ serves as the "probability measure" on A. Given two random variables $a_1, a_2 \in A$ we say that a_1 and a_2 are free wrt. ϕ if for all $n \in \mathbb{N}$ and all polynomials $(p_i)_{i=1}^n$ we have

$$\phi(p_1(a_{i_1})p_2(a_{i_2})\cdots p_n(a_{i_n})) = 0,$$

whenever $i_1 \neq i_2 \neq \cdots i_n \neq i_1$ and $\phi(p_j(a_j)) = 0$ for $j = 1, \dots n$.

It turns out that freeness of a_1 and a_2 and knowledge of $\phi(a_1^k)$ and $\phi(a_2^k)$ for all $k \in \mathbb{N}$ is enough to compute the distributions of a_1+a_2 and a_1a_2, i.e. finding the moments $\phi((a_1+a_2)^k)$ and $\phi((a_1a_2)^k)$ respectively for all $k \in \mathbb{N}$. This is done in turn by use of Voiculescu's R- and S-transforms.

For fixed $a \in A$ one define the Cauchy-tranform of a by

$$G_a(z) = \phi((z-a)^{-1})$$

for $z \in \mathbb{C}$ whenever this makes sense, and also define

$$\psi_a(z) = \phi((1-za)^{-1}) - 1$$

for $z \in \mathbb{C}$. The R- and S-transforms are then defined by

$$R_a(z) = G_a^{\langle -1 \rangle}(z) - z^{-1}$$

and

$$S_a(z) = \frac{1+z}{z}\psi_a^{\langle -1 \rangle}(z)$$

for $z \in \mathbb{C}$, where superscript $\langle -1 \rangle$ denotes inversion wrt. composition. Voiculescu then proved the following formulas [Voi1], [Voi2]:

$$R_{a_1+a_2}(z) = R_{a_1}(z) + R_{a_2}(z) \tag{1.1}$$

and

$$S_{a_1 \cdot a_2}(z) = S_{a_1}(z) \cdot S_{a_2}(z). \tag{1.2}$$

when a_1 and a_2 are free wrt. ϕ and $z \in \mathbb{C}$ is chosen suitably.

Alternative proofs of the above formulas can be found in [Haa], [Sp1] and [NS1]. A nice combinatorial review can be found in [Sp3].

Instead of considering a "classical" free probability space one can consider the more general notion of an amalgamated free probability space, i.e. let $\mathcal{A}$ be a unital Banach algebra, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital Banach sub-algebra of $\mathcal{A}$, and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation. Thus E is linear, $E(b) = b$ for all $b \in \mathcal{B}$, E is norm-decreasing, and E has the $\mathcal{B}$-bimodule property;

$$E(b_1 a b_2) = b_1 E(a) b_2$$

for $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$. We will say that $(\mathcal{B} \subset \mathcal{A}, E)$ is a $\mathcal{B}$-probability space.

By replacing freeness above by the corresponding freeness with amalgamation Voiculescu proved an amalgamated version of (1.1) in [Voi3], and Speicher reproved this by combinatorial means in [Sp2]. The definition of freeness wih amalgamation is as follows. Let $a_1, a_2 \in \mathcal{A}$ be random variables in $\mathcal{A}$. We will say that a_1 and a_2 are free with amalgamation over $\mathcal{B}$ wrt. E (or free wrt. $\mathcal{B}$ or simply $\mathcal{B}$-free) if for all $n \in \mathbb{N}$ and for all $x_j \in \operatorname{alg}(\mathcal{B}, a_{i_j})$ we have

$$E(x_1 x_2 \cdots x_n) = 0, \tag{1.3}$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$ and $E(x_j) = 0$ for all $j = 1, \ldots, n$. In other words the indices has to be alternating since we are only dealing with two random variables.

Note, that if the condition $E(x_j) = 0$, $1 \leq j \leq n$ is relaxed to $E(x_j) = 0$ for $2 \leq j \leq n-1$, (1.3) should be changed to

$$E(x_1 x_2 \cdots x_{n-1}) = \begin{cases} E(x_1)E(x_{n-1}), & n = 2 \\ 0, & n \geq 3 \end{cases}. \tag{1.4}$$

This can be seen by writing $x_1 = x_1^0 + E(x_1)$ and $x_n = x_n^0 + E(x_n)$, where $E(x_1^0) = E(x_n^0) = 0$.

In the unital Banach-algebra seting we give a new proof of the additiveness property of the amalgamated R-transform. We also prove an amalgamated version of (1.2) for the amalgamated S-transform in

the case where $\mathcal{B}$ is abelian. Our methods are strongly inspired by section 3 of [Haa] and as in [Haa] we give concrete neighboorhoods where amalgamated versions of (1.1) and (1.2) are valid.

2. Amalgamated R-transform in Banach-algebras

Throughout this paper we let $\mathcal{A}$ be a unital Banach algebra, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital Banach sub-algebra of $\mathcal{A}$, and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation, so that $(\mathcal{B} \subset \mathcal{A}, E)$ is a $\mathcal{B}$-probability space.

For $\epsilon > 0$ we denote by $\mathcal{B}(0,\epsilon)$ the ϵ-neighborhood of 0 in $\mathcal{B}$ and by $\mathcal{B}(0,\epsilon)_{\text{inv}}$ the invertible elements in $\mathcal{B}$ of norm strictly less than ϵ. Finally $\overline{\mathcal{B}}(0,\epsilon)$ will denote the norm-closure of $\mathcal{B}(0,\epsilon)$.

Let $a \in \mathcal{A}$ be fixed. Define the function $g_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ by

$$g_a(b) := bE((1-ab)^{-1}) = E((1-ba)^{-1})b \tag{2.1}$$

for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$. For $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ observe that by the Carl Neumann series $g_a(b)$ is just the absolute convergent sum

$$g_a(b) = b + bE(a)b + bE(aba)b + \cdots .$$

Of course, if $b \in \mathcal{B}(0, \frac{1}{\|a\|})_{\text{inv}}$ then $g_a(b) = E((b^{-1}-a)^{-1})$ is nothing but $G_a(b^{-1})$, where G_a is the amalgamated Cauchy-transform of a.

Lemma 2.1. *Let $g_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ be the function defined by (2.1). Then g_a is 1-1 on $\mathcal{B}(0, \frac{1}{4\|a\|})$.*

Proof. Let $b_1, b_2 \in \mathcal{B}$ such that $\|b_i\| < \frac{\alpha}{\|a\|}$ for $0 < \alpha < 1$ to be determined. The idea is to determine α such that

$$\left\| \big((g_a(b_1) - g_a(b_2)\big) - \big(b_1 - b_2\big) \right\| < \|b_1 - b_2\| \tag{2.2}$$

for $\|b_1\|, \|b_2\| < \frac{\alpha}{\|a\|}$.

Observe that

$$\begin{aligned} g_a(b_1) - g_a(b_2) &= (b_1 - b_2) + E(b_1ab_1) - E(b_2ab_2) \\ &\qquad + E(b_1ab_1ab_1) - E(b_2ab_2ab_2) + \cdots \\ &= (b_1 - b_2) + E((b_1 - b_2)ab_1) + E(b_2a(b_1 - b_2)) \\ &\qquad + E(b_1 - b_2)ab_1ab_1) + E(b_2a(b_1 - b_2)ab_2) \\ &\qquad\qquad + E(b_2ab_2a(b_1 - b_2)) + \cdots , \end{aligned}$$

so we can estimate the left-hand side of (2.2) by

$$\begin{aligned}\|g_a(b_1) - g_a(b_2) - (b_1 - b_2)\| &\leq 2\|a\| \max\{\|b_1\|, \|b_2\|\} \|b_1 - b_2\| \\ &\quad + 3\|a\|^2 \max\{\|b_1\|, \|b_2\|\}^2 \|b_1 - b_2\| + \cdots \\ &\leq \|b_1 - b_2\| \left(\sum_{n=0}^{\infty} (n+1)\alpha^n - 1 \right) \\ &= \left(\frac{1}{(1-\alpha)^2} - 1 \right) \|b_1 - b_2\| .\end{aligned}$$

We infer that if $\alpha < \frac{1}{4}$ then (2.2) is satisfied. □

We are interested in a neighborhood $\mathcal{B}(0, \frac{\beta}{\|a\|})$, where $0 < \beta < 1$ is to be determined, such that g_a maps $\mathcal{B}(0, \frac{1}{4\|a\|})$ onto a neighborhood of 0 which contains the neighborhood $\mathcal{B}(0, \frac{\beta}{\|a\|})$. For this we need the following lemma that enables us to use the Inverse Function Theorem [VLH, Theorem 3.6.3] on g_a.

Lemma 2.2. *Let $a \in \mathcal{A}$ be fixed. Define $g_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ by $g_a(b) = bE((1-ab)^{-1})$ for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$. Then g_a is Fréchet differentiable, and the differential of g_a is*

$$Dg_a(b) : h \mapsto E((1-ba)^{-1}h(1-ab)^{-1})$$

for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ and $h \in \mathcal{B}$. Furthermore the differential, $Dg_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$, is continuous, so that g_a is differentiable of class C^1. If $b \in \mathcal{B}(0, \frac{\alpha}{\|a\|})$ for $0 < \alpha < 1$ then

$$\|Dg_a(b) - Dg_a(0)\| < \frac{\alpha(2-\alpha)}{(1-\alpha)^2}. \tag{2.3}$$

Proof. If $b_1, b_2 \in \mathcal{B}(0, \frac{1}{\|a\|})$ then

$$\begin{aligned}&\left\|(1-b_1a)^{-1} - (1-b_2a)^{-1}\right\| \\ &\quad= \left\|(1-b_1a)^{-1}\big((1-b_2a) - (1-b_1a)\big)(1-b_2a)^{-1}\right\| \\ &\quad= \left\|(1-b_1a)^{-1}(b_2 - b_1)a(1-b_2a)^{-1}\right\| \\ &\quad\leq \left\|(1-b_1a)^{-1}\right\| \left\|(1-b_2a)^{-1}\right\| \|a\| \|b_2 - b_1\| \\ &\quad\leq \frac{1}{1 - \|a\| \|b_1\|} \frac{1}{1 - \|b_2\| \|a\|} \|a\| \|b_2 - b_1\| \to 0\end{aligned} \tag{2.4}$$

as $b_2 \to b_1$ in norm. Also

$$\begin{aligned}(2.5)\quad g_a(b_1) - g_a(b_2) &= E((1-b_1a)^{-1}b_1 - b_2(1-ab_2)^{-1}) \\ &= E\left((1-b_1a)^{-1}\big(b_1(1-ab_2) - (1-b_1a)b_2\big)(1-ab_2)^{-1}\right) \\ &= E((1-b_1a)^{-1}(b_1-b_2)(1-ab_2)^{-1}).\end{aligned}$$

Combining (2.4) and (2.5) we conclude that for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ fixed, and $h \in \mathcal{B}$ of small norm the first part of the lemma now follows since

$$\begin{aligned}g_a(b+h) - g_a(b) &= E((1-(b+h)a)^{-1}h(1-ab)^{-1}) \\ &= E((1-ba)^{-1}h(1-ab)^{-1}) \\ &\quad + E\left(\left((1-(b+h)a)^{-1} - (1-ba)^{-1}\right)h(1-ab)^{-1}\right) \\ &= E((1-ba)^{-1}h(1-ab)^{-1}) + O(\|h\|^2).\end{aligned}$$

To see that $Dg_a : b \mapsto Dg_a(b)$ is continuous for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ we observe that if $b_1, b_2 \in \mathcal{B}(0, \frac{1}{\|a\|})$ then (2.4) implies

$$\begin{aligned}(2.6)\quad \|Dg_a(b_1) - Dg_a(b_2)\| &= \sup_{\|h\|\le 1} \|Dg_a(b_1)(h) - Dg_a(b_2)(h)\| \\ &= \sup_{\|h\|\le 1} \left\|E((1-b_1a)^{-1}h(1-ab_1)^{-1}) - E((1-b_2a)^{-1}h(1-ab_2)^{-1}))\right\| \\ &\le \sup_{\|h\|\le 1} \left\|(1-b_1a)^{-1}h(1-ab_1)^{-1} - (1-b_1a)^{-1}h(1-ab_2)^{-1})\right\| \\ &\quad + \sup_{\|h\|\le 1} \left\|(1-b_1a)^{-1}h(1-ab_2)^{-1} - (1-b_2a)^{-1}h(1-ab_2)^{-1})\right\| \\ &\le \left\|(1-b_1a)^{-1}\right\| \left\|(1-ab_1)^{-1} - (1-ab_2)^{-1}\right\| \\ &\quad + \left\|(1-b_1a)^{-1} - (1-b_2a)^{-1}\right\| \left\|(1-ab_2)^{-1}\right\| \to 0\end{aligned}$$

as $b_2 \to b_1$ in norm.

Let $0 < \alpha < 1$. For $b \in \mathcal{B}(0, \frac{\alpha}{\|a\|})$ we use (2.6) (with $b_1 = b$ and $b_2 = 0$) to see that

$$\begin{aligned}\|Dg_a(b) - Dg_a(0)\| &\le \left\|(1-ba)^{-1}\right\| \left\|(1-ab)^{-1} - 1\right\| \\ &\quad + \left\|(1-ba)^{-1} - 1\right\| \\ &\le \left(\frac{1}{1-\|b\|\,\|a\|} + 1\right)\left(\sum_{n=1}^{\infty} \|a\|^n \|b\|^n\right) \\ &= \left(\frac{1}{1-\alpha} + 1\right)\frac{\alpha}{1-\alpha} = \frac{\alpha(2-\alpha)}{(1-\alpha)^2}.\end{aligned}$$

□

Proposition 2.3. *Let $\mathcal{A}$ be a unital Banach algebra, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital Banach sub-algebra, and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation. Let $a \in \mathcal{A}$ be a fixed element. The function $g_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ defined by $g_a(b) = bE((1-ab)^{-1})$ for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ is a bijection of the neighborhood $\mathcal{B}(0, \frac{1}{4\|a\|})$ onto a neighborhood of 0 which contains $\mathcal{B}(0, \frac{1}{11\|a\|})$. Furthermore*

$$g_a^{<-1>}\left(\mathcal{B}(0, \tfrac{1}{11\|a\|})\right) \subseteq \mathcal{B}(0, \tfrac{2}{11\|a\|}) \tag{2.7}$$

$$g_a^{<-1>}\left(\mathcal{B}(0, \tfrac{1}{11\|a\|})_{\mathrm{inv}}\right) \subseteq \mathcal{B}(0, \tfrac{2}{11\|a\|})_{\mathrm{inv}} \tag{2.8}$$

The proof is actually just an inspection of some of the proof of the Inverse Function Theorem, and can be found in any standard text book on the subject. The proof we inspect here is taken from [VLH]. Since the estimates are of importance to us we include a proof.

Proof. Define $T : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ by $T(b) = b - g_a(b)$ for $b \in B(0, \frac{1}{\|a\|})$, and observe that $T(0) = 0$. By lemma 2.2 we have

$$DT(0) = D\,\mathrm{id}_{\mathcal{B}}(0) - Dg_a(0) = D\,\mathrm{id}_{\mathcal{B}}(0) - D\,\mathrm{id}_{\mathcal{B}}(0) = 0.$$

By lemma 2.2 Dg_a is continuous so DT is also continuous, and

$$\|DT(b)\| = \|Dg_a(b) - Dg_a(0)\| < \frac{\alpha(2-\alpha)}{(1-\alpha)^2}$$

for $b \in \mathcal{B}(0, \frac{\alpha}{\|a\|})$ and $0 < \alpha < 1$. If we choose $\alpha = \frac{2+\epsilon}{11}$ for $\epsilon > 0$ small enough we have

$$\|DT(b)\| < \frac{\frac{2+\epsilon}{11}(2 - \frac{2+\epsilon}{11})}{(1 - \frac{2+\epsilon}{11})^2} < \frac{1}{2}.$$

By the Mean Value Theorem [VLH] we conclude that

$$\|T(b)\| \leq \frac{1}{2}\|b\|$$

for $b \in \mathcal{B}(0, \frac{2+\epsilon}{11\|a\|})$.

Now let $\tilde{b} \in \mathcal{B}(0, \frac{2+\epsilon}{22\|a\|})$ and define

$$T_{\tilde{b}}(b) = \tilde{b} - T(b) = \tilde{b} + b - g_a(b)$$

for $b \in \overline{\mathcal{B}}(0, \frac{2+\epsilon}{11\|a\|})$. For $b \in \overline{\mathcal{B}}(0, \frac{2+\epsilon}{11\|a\|})$ we have

$$\|T_{\tilde{b}}(b)\| = \|\tilde{b} - T(b)\| \leq \|\tilde{b}\| + \|T(b)\| \leq \tfrac{2+\epsilon}{11\|a\|},$$

so actually $T_{\tilde{b}}$ is a continuous map from the Banach space $\overline{\mathcal{B}}(0, \frac{2+\epsilon}{11\|a\|})$ into itself. Also $T_{\tilde{b}} : \overline{\mathcal{B}}(0, \frac{2+\epsilon}{11\|a\|}) \to \overline{\mathcal{B}}(0, \frac{2+\epsilon}{11\|a\|})$ is a contraction since for $b_1, b_2 \in \overline{\mathcal{B}}(0, \frac{2+\epsilon}{11\|a\|})$ we have

$$\|T_{\tilde{b}}(b_1) - T_{\tilde{b}}(b_2)\| = \|T(b_1) - T(b_2)\| \leq \tfrac{1}{2} \|b_1 - b_2\|,$$

again by the Mean Value Theorem [VLH]. Banach's Fix-point Theorem [VLH] now implies that there exists a unique fix-point $b \in \overline{\mathcal{B}}(0, \frac{2+\epsilon}{11\|a\|})$ of $T_{\tilde{b}}$. Thus $g_a(b) = \tilde{b}$, so we have shown that

$$\overline{\mathcal{B}}(0, \tfrac{2+\epsilon}{22\|a\|}) \subseteq g_a(\overline{\mathcal{B}}(0, \tfrac{2+\epsilon}{11\|a\|})).$$

Of course an argument similar to the above also shows that

$$\overline{\mathcal{B}}(0, \tfrac{1}{11\|a\|}) \subseteq g_a(\overline{\mathcal{B}}(0, \tfrac{2}{11\|a\|})). \tag{2.9}$$

Now assume that $\|b\| = \frac{2}{11\|a\|}$. If $\|g_a(b)\| < \frac{1}{11\|a\|}$ then since g_a is continuous we could also find a $b_1 \in \mathcal{B}$ such that $\frac{2}{11\|a\|} < \|b_1\| < \frac{2+\epsilon}{11\|a\|}$ and such that $\|g_a(b_1)\| < \frac{1}{11\|a\|}$. But by (2.9) we could also find a $b_2 \in \overline{\mathcal{B}}(0, \frac{2}{11\|a\|})$ such that $g_a(b_2) = g_a(b_1)$ thus contradicting the injectivity of g_a on $\mathcal{B}(0, \frac{1}{4\|a\|})$ that follows from lemma 2.1. We have now shown that g_a is a bijection of the neighborhood $\mathcal{B}(0, \frac{2}{11\|a\|})$ onto a neighborhood of 0 which contains $\mathcal{B}(0, \frac{1}{11\|a\|})$ and this is exactly (2.7).

To prove (2.8), define $\Psi_a(b) := \sum_{n=1}^{\infty} E((ab)^n)$ for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$. For $b \in \mathcal{B}(0, \frac{1}{2\|a\|})$ we have

$$\|\Psi_a(b)\| \leq \sum_{n=1}^{\infty} (\|a\| \, \|b\|)^n < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

so $1 + \Psi_a(b)$ is invertible. Thus if $b \in \mathcal{B}(0, \frac{1}{2\|a\|})$ we have

$$g_a(b) = bE((1 - ab)^{-1}) = b(1 + \Psi_a(b)),$$

so invertability of $g_a(b)$ is equivalent to invertability of b and thus (2.8) follows. $\square$

Proposition 2.3 now assures well-definedness when we to define Voiculescu's amalgamated R-transform in the Banach algebra setting.

Definition 2.4. Let $\mathcal{A}$ be a unital Banach algebra, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital Banach sub-algebra, and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation. Let $a \in \mathcal{A}$ be a fixed non-zero element. Then the amalgamated R-transform of a is defined by

$$R_a(b) := \left(g_a^{<-1>}(b)\right)^{-1} - b^{-1} \tag{2.10}$$

for $b \in \mathcal{B}(0, \frac{1}{11\|a\|})_{\text{inv}}$.

Again we can also use proposition 2.3 to show the additiveness of the amalgamated R-transform on free random variables.

Theorem 2.5. *Let $\mathcal{A}$ be a unital Banach algebra, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital Banach sub-algebra, and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation. Let $a_1, a_2 \in \mathcal{A}$ be two $\mathcal{B}$-free random variables with respect to E. Let $b \in \mathcal{B}(0, \min(\frac{1}{11\|a_1+a_2\|}, \frac{1}{11\|a_1\|}, \frac{1}{11\|a_2\|}))_{\text{inv}}$. Then the amalgamated R-transform satisfies*

$$R_{a_1+a_2}(b) = R_{a_1}(b) + R_{a_2}(b).$$

Proof. In the following proof i always denotes an index such that $i \in \{1,2\}$. Recall that $g_{a_i}^{<-1>}$ is defined and one-to-one in $B(0, \frac{1}{11\|a_i\|})_{\text{inv}}$ by proposition 2.3. Let $b \in \mathcal{B}(0, \min(\frac{1}{11\|a_1+a_2\|}, \frac{1}{11\|a_1\|}, \frac{1}{11\|a_2\|}))_{\text{inv}}$ and let $b_1, b_2 \in \mathcal{B}$ be the uniquely determined elements that satisfy $b_i \in \mathcal{B}(0, \frac{2}{11\|a_i\|})_{\text{inv}}$ and

$$b = g_{a_i}(b_i), \tag{2.11}$$

for $i \in \{1,2\}$. Observe that

$$b_i^{-1}b = b_i^{-1}b_iE((1-a_ib_i)^{-1}) = 1 + \sum_{n=1}^{\infty} E((a_ib_i)^n),$$

and thus

$$b_1^{-1}b + b_2^{-1}b - 1 = 1 + \sum_{n=1}^{\infty} E((a_1b_1)^n) + \sum_{n=1}^{\infty} E((a_2b_2)^n),$$

so we infer that $b_1^{-1}b + b_2^{-1}b - 1$ is invertible and that

$$\left\|(b_1^{-1}b + b_2^{-1}b - 1)^{-1}\right\| \leq 1 + 2\frac{\frac{2}{11}}{1-\frac{2}{11}} < 2,$$

because $b_i \in \mathcal{B}(0, \frac{2}{11\|a_i\|})$. If we define $b_3 = b(b_1^{-1}b + b_2^{-1}b - 1)^{-1}$ then obviously $b_3 \in \mathcal{B}(0, \frac{2}{11\|a_1+a_2\|})_{\text{inv}}$ because $b \in \mathcal{B}(0, \frac{1}{11\|a_1+a_2\|})_{\text{inv}}$. Furthermore

$$b_3^{-1} = b_1^{-1} + b_2^{-1} - b^{-1}. \tag{2.12}$$

Define $A_1(b_1) = (b_1^{-1} - a_1)^{-1} - b$ and $A_1(b_2) = (b_2^{-1} - a_2)^{-1} - b$. By (2.11) we have $E(A_1(b_1)) = E(A_2(b_2)) = 0$. Note that

$$\begin{aligned}
&(b_1^{-1} - a_1)b\left(1 - b^{-1}A_1(b_1)b^{-1}A_2(b_2)\right)(b_2^{-1} - a_2)\\
&= (b_1^{-1} - a_1)\left(b - ((b_1^{-1} - a_1)^{-1} - b)b^{-1}((b_2^{-1} - a_2)^{-1} - b)\right)(b_2^{-1} - a_2)\\
&\qquad = (b_1^{-1} - a_1)\Big(b - (b_1^{-1} - a_1)^{-1}b^{-1}(b_2^{-1} - a_2)^{-1}\\
&\qquad\quad + (b_1^{-1} - a_1)^{-1} + (b_2^{-1} - a_2)^{-1} - b\Big)(b_2^{-1} - a_2)\\
&\qquad\qquad = -b^{-1} + (b_1^{-1} - a_1) + (b_2^{-1} - a_2)\\
&\qquad\qquad\qquad = b_3^{-1} - (a_1 + a_2).
\end{aligned}$$

Inverting we have

$$\begin{aligned}
(2.13)\quad &(b_3^{-1} - (a_1 + a_2))^{-1}\\
&= (b_2^{-1} - a_2)^{-1}(1 - b^{-1}A_1(b_1)b^{-1}A_2(b_2))^{-1}b^{-1}(b_1^{-1} - a_1)^{-1}.
\end{aligned}$$

We want to use the Carl Neumann series in (2.13) so we observe that

$$\begin{aligned}
b^{-1}A_i(b_i) &= b^{-1}((b_i^{-1} - a_i)^{-1} - b)\\
&= (b_iE((1 - a_ib_i)^{-1})))^{-1}b_i(1 - a_ib_i)^{-1} - 1\\
&= \left(1 + \sum_{n=1}^{\infty} E((a_ib_i)^n)\right)^{-1}\left(1 + \sum_{n=1}^{\infty}(a_ib_i)^n\right) - 1\\
&= \left(1 + \sum_{k=1}^{\infty}\left(-\sum_{n=1}^{\infty} E((a_ib_i)^n)\right)^k\right)\left(1 + \sum_{n=1}^{\infty}(a_ib_i)^n\right) - 1\\
&= \sum_{k=1}^{\infty}\left(-\sum_{n=1}^{\infty} E((a_ib_i)^n)\right)^k + \sum_{n=1}^{\infty}(a_ib_i)^n\\
&\quad + \left(\sum_{k=1}^{\infty}\left(-\sum_{n=1}^{\infty} E((a_ib_i)^n)\right)^k\right)\left(\sum_{n=1}^{\infty}(a_ib_i)^n\right).
\end{aligned}$$

We infer that

$$(2.14)\qquad \left\|b^{-1}A_i(b_i)\right\| \le \frac{\frac{2}{9}}{1 - \frac{2}{9}} + \frac{\frac{2}{11}}{1 - \frac{2}{11}} + \frac{\frac{2}{9}}{1 - \frac{2}{9}}\frac{\frac{2}{11}}{1 - \frac{2}{11}} < 1.$$

The Carl Neumann series now applies to (2.13):

$$(b_3^{-1}-(a_1+a_2))^{-1} = (b_2^{-1}-a_2)^{-1}\left(\sum_{n=0}^{\infty}\big(b^{-1}A_1(b_1)b^{-1}A_2(b_2)\big)^n\right)b^{-1}(b_1^{-1}-a_1)^{-1} \tag{2.15}$$

Since $A_1(b_1)$ and $A_2(b_2)$ are $\mathcal{B}$-free and centered (2.15) and (1.4) implies

$$\begin{aligned} g_{a_1+a_2}(b_3) &= E((b_3^{-1}-(a_1+a_2))^{-1}) \\ &= E((b_2^{-1}-a_2)^{-1}b^{-1}(b_1^{-1}-a_1)^{-1}) \\ &= E((b_2^{-1}-a_2)^{-1})b^{-1}E((b_1^{-1}-a_1)^{-1}) \\ &= bb^{-1}b = b. \end{aligned}$$

Thus by (2.12)

$$\begin{aligned} R_{a_1+a_2}(b) &= (g_{a_1+a_2}^{<-1>}(b))^{-1}-b^{-1} \\ &= b_3^{-1}-b^{-1} \\ &= (b_1^{-1}-b^{-1})+(b_2^{-1}-b^{-1}) \\ &= (g_{a_1}^{<-1>}(b))^{-1}-b^{-1}+(g_{a_2}^{<-1>}(b))^{-1}-b^{-1} \\ &= R_{a_1}(b)+R_{a_2}(b). \end{aligned}$$

□

3. Connection to Speicher's combinatorial approach

Actually we do not have to restrict ourselves to invertible elements in definition 2.4, because we will now show that the R-transform has a $\mathcal{B}$-removable singularity in each non-invertible element exactly as is the case for the scalar R-transform [Haa, prop. 3.1].

For this we need the following description of the sum over all irreducible non crossing partitions. We wil say that a partition $\pi \in \mathrm{NC}(r)$ is irreducible if $1 \sim_\pi r$ and we will denote concatenation of non-crossing partitions by $\sqcup$.

Lemma 3.1. *let $\mathcal{A}$ be a unital Banach algebra, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital Banach sub-algebra of $\mathcal{A}$, and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation. Then*

$$\sum_{j=1}^{r}(-1)^{j+1}\sum_{\substack{n_1+\cdots+n_j=r\\ n_1,\dots,n_j\ge 1}}\ \sum_{\substack{\pi\in\mathrm{NC}(r)\\ \pi\le 1_{n_1}\sqcup\cdots\sqcup 1_{n_j}}}\kappa_\pi^{\mathcal{B}}((ba)^{\otimes r}) = \sum_{\substack{\pi\in\mathrm{NC}(r)\\ 1\sim_\pi r}}\kappa_\pi^{\mathcal{B}}((ba)^{\otimes r}), \tag{3.1}$$

for all $r \in \mathbb{N}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Proof. The proof is on induction in r. For $r = 1$ (3.1) is obvious since both sides of the equation reduces to $\kappa_1^{\mathcal{B}}(ba)$.

Now assume that (3.1) is true for all indices strictly less that r. Then the left-hand side of (3.1) is

$$E((ba)^r)+\sum_{j=2}^{r}(-1)^{j+1}\sum_{n_1=1}^{r-j+1}\sum_{\substack{n_2+\cdots+n_j=r-n_1\\ n_2,\ldots,n_j\geq 1}}\sum_{\substack{\pi\in\mathrm{NC}(r)\\ \pi\leq 1_{n_1}\sqcup\cdots\sqcup 1_{n_j}}}\kappa_\pi^{\mathcal{B}}((ba)^{\otimes r}). \tag{3.2}$$

Since "$\sum_{j=2}^{r}\sum_{n_1=1}^{r-j+1}=\sum_{n_1=1}^{r-1}\sum_{j=2}^{r-n_1+1}$" we can rewrite (3.2) as

$$E((ba)^r)+\left(\sum_{n_1=1}^{r-1}\sum_{\pi_1\in\mathrm{NC}(n_1)}\kappa_{\pi_1}^{\mathcal{B}}((ba)^{\otimes n_1})\sum_{j=2}^{r-n_1+1}(-1)^{j+1}\right.$$
$$\left.\sum_{\substack{n_2+\cdots+n_j=r-n_1\\ n_2,\ldots,n_j\geq 1}}\sum_{\substack{\pi\in\mathrm{NC}(r-n_1)\\ \pi\leq 1_{n_2}\sqcup\cdots\sqcup 1_{n_j}}}\kappa_\pi^{\mathcal{B}}((ba)^{\otimes(r-n_1)})\right).$$

Defining index $i := j-1$ and $m_l := n_{l+1}$ for $l\in\{1,\ldots,j-1\}$ we have

$$E((ba)^r)+\left(\sum_{n_1=1}^{r-1}\sum_{\pi_1\in\mathrm{NC}(n_1)}\kappa_{\pi_1}^{\mathcal{B}}((ba)^{\otimes n_1})\right. \tag{3.3}$$
$$\left.\left(-\sum_{i=1}^{r-n_1}(-1)^{i+1}\sum_{\substack{m_1+\cdots+m_i=r-n_1\\ m_1,\ldots,m_i\geq 1}}\sum_{\substack{\pi\in\mathrm{NC}(r-n_1)\\ \pi\leq 1_{m_1}\sqcup\cdots\sqcup 1_{m_i}}}\kappa_\pi^{\mathcal{B}}((ba)^{\otimes(r-n_1)})\right)\right).$$

Using the induction hypotheses on the inner parenthesis of (3.3) we have

$$E((ba)^r) \tag{3.4}$$
$$-\left(\sum_{n_1=1}^{r-1}\sum_{\pi_1\in\mathrm{NC}(n_1)}\kappa_{\pi_1}^{\mathcal{B}}((ba)^{\otimes n_1})\left(\sum_{\substack{\pi\in\mathrm{NC}(n_1+1,\ldots,r)\\ (n_1+1)\sim_\pi r}}\kappa_\pi^{\mathcal{B}}((ba)^{\otimes(r-n_1)})\right)\right).$$

But the (outer) parenthesis in (3.4) is just the sum over all non-irreducible partitions in $\mathrm{NC}(r)$, so (3.4) reduces to

$$\sum_{\substack{\pi\in\mathrm{NC}(r)\\ 1\sim_\pi r}}\kappa_\pi^{\mathcal{B}}((ba)^{\otimes r}).$$

□

We are now ready to remove the $\mathcal{B}$-singularities of the R-transform. Let $g_a(b) \in \mathcal{B}(0, \frac{1}{11\|a\|})_{\text{inv}}$ for some $b \in \mathcal{B}(0, \frac{2}{11\|a\|})_{\text{inv}}$. Then

$$\begin{aligned}
R_a(g_a(b)) &= b^{-1} - (g_a(b))^{-1} \\
&= b^{-1} - b^{-1}(E(1-ba)^{-1})^{-1} \\
&= b^{-1} - b^{-1}\left(1 - \left(-\sum_{n=1}^{\infty} E((ba)^n)\right)\right)^{-1} \\
&= b^{-1}\sum_{j=1}^{\infty}(-1)^{j+1}\left(\sum_{n=1}^{\infty} E((ba)^n)\right)^{j} \\
&= \left(\sum_{n=0}^{\infty} E(a(ba)^n)\right)\left(1 + \sum_{n=1}^{\infty} E((ba)^n)\right)^{-1} \\
&= E(a(1-ba)^{-1})(E(1-ba)^{-1})^{-1}. \qquad (3.5)
\end{aligned}$$

We can thus adopt (3.5) as the definition of the R-transform even for non-invertible $g_a(b) \in \mathcal{B}(0, \frac{1}{11\|a\|})$. Doing this we recover Speicher's definition of the amalgamated R-transform from [Sp2, Th. 4.1.12] by use of lemma 3.1. We have

$$\begin{aligned}
R_a(g_a(b)) &= \sum_{n=0}^{\infty} E(a(ba)^n)\left(1 + \sum_{n=1}^{\infty} E((ba)^n)\right)^{-1} \\
&= \sum_{n=0}^{\infty} E(a(ba)^n)\left(\sum_{j=1}^{\infty}(-1)^{j+1}\left(\sum_{n=1}^{\infty} E((ba)^n)\right)^{j-1}\right) \\
&= b^{-1}\sum_{r=1}^{\infty}\left(\sum_{j=1}^{r}(-1)^{j+1}\sum_{\substack{n_1+\dots+n_j=r \\ n_1,\dots,n_j\geq 1}} E((ba)^{n_1})\cdots E((ba)^{n_j})\right) \qquad (3.6)
\end{aligned}$$

Actually the last line does not make sense for non-invertible elements, but the singularity is obviously removable because E has the $\mathcal{B}$-bi-module-property, and the following computations becomes notationally simpler, when we write (3.6) in this way. We now use lemma 3.1 in

(3.6)

$$R_a(g_a(b)) = b^{-1}\sum_{r=1}^{\infty}\left(\sum_{j=1}^{r}(-1)^{j+1}\sum_{\substack{n_1+\cdots+n_j=r\\ n_1,\ldots,n_j\geq 1}}\sum_{\substack{\pi\in\mathrm{NC}(r)\\ \pi\leq 1_{n_1}\sqcup\cdots\sqcup 1_{n_j}}}\kappa_\pi^{\mathcal{B}}((ba)^{\otimes r})\right)$$

$$(3.7) \qquad = b^{-1}\sum_{r=1}^{\infty}\left(\sum_{\substack{\pi\in\mathrm{NC}(r)\\ 1\sim_\pi r}}\kappa_\pi^{\mathcal{B}}((ba)^{\otimes r})\right)$$

Rearranging the sum after the number of elements in the block that contains 1 and r (which are always in the same block), we actually just sum over all possible outer partitions.

(3.8) $R_a(g_a(b)) =$

$$\sum_{r=2}^{\infty}\sum_{j=2}^{r}\sum_{\substack{i_2+i_3+\cdots i_j+j=r\\ i_2,i_3,\ldots,i_j\geq 0}}\kappa_\pi^{\mathcal{B}}\left(a\otimes E((ba)^{i_2})ba\otimes E((ba)^{i_3})ba\otimes\cdots E((ba)^{i_j})ba\right)$$

$$=\sum_{k=2}^{\infty}\sum_{i_2,i_3,\ldots,i_k=0}^{\infty}\kappa_\pi^{\mathcal{B}}\left(a\otimes E((ba)^{i_2})ba\otimes E((ba)^{i_3})ba\otimes\cdots E((ba)^{i_k})ba\right).$$

Rearranging terms in (3.8) we get

$$R_a(g_a(b)) = \sum_{r=1}^{\infty}\kappa_r^{\mathcal{B}}\left(a\otimes\left(\left(\sum_{n=0}^{\infty}E((ba)^n)\right)ba\right)^{\otimes(r-1)}\right)$$

$$(3.9) \qquad = \sum_{r=1}^{\infty}\kappa_r^{\mathcal{B}}(a\otimes g_a(b)a\otimes\cdots\otimes g_a(b)a).$$

This is exactly Speicher's way of defining the amalgamated R-transform, and additivity of the R-transform on $\mathcal{B}$-free variables thus follows from [Sp2, th. 4.1.7]. The new thing is that we have produced a concrete neighborhood where the sum in (3.9) makes sense, that is, if $g_a(b)\in\mathcal{B}(0,\frac{1}{11\|a\|})$ for some $b\in\mathcal{B}(0,\frac{2}{11\|a\|})$ then (3.9) is convergent.

4. Amalgamated S-transform in Banach-algebras

Again we let $\mathcal{A}$ be a unital Banach algebra, $1\in\mathcal{B}\subset\mathcal{A}$ be a unital Banach sub-algebra of $\mathcal{A}$ and $E:\mathcal{A}\to\mathcal{B}$ a conditional expectation. The results on the amalgamated S-transform are obtained via a similar approach as in the last section.

Let $a \in \mathcal{A}$ be a fixed element. Define $\Psi_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ by

$$\Psi_a(b) = \sum_{n=1}^{\infty} E((ba)^n) = E((1-ba)^{-1}) - 1, \tag{4.1}$$

for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$.

We proceed as in the previous section to show that Ψ_a is injective in a neighborhood $\mathcal{B}(0, \frac{\alpha}{\|a\|})$ and maps $\mathcal{B}(0, \frac{\alpha}{\|a\|})$ onto a neighborhood of 0 which contains a neighborhood $\mathcal{B}(0, \frac{\beta}{\|a\|})$, where α and β are constants to be determined.

Lemma 4.1. *Let $a \in \mathcal{A}$ such that $E(a) \in \mathcal{B}_{\mathrm{inv}}$ and let $\Psi_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ be the function defined by (4.1). Then Ψ_a is 1-1 on $\mathcal{B}(0, \frac{1}{4\|a\|^2\|E(a)^{-1}\|})$.*

Proof. Define $\Gamma_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ by $\Gamma_a : b \mapsto \Psi_a(b)E(a)^{-1}$. Let $b_1, b_2 \in \mathcal{B}(0, \frac{\alpha}{\|a\|^2\|E(a)^{-1}\|})$ where $0 < \alpha < 1$ is to be determined. Observe that since $1 \leq \|a\| \|E(a)^{-1}\|$ we have $\mathcal{B}(0, \frac{\alpha}{\|a\|^2\|E(a)^{-1}\|}) \subseteq \mathcal{B}(0, \frac{\alpha}{\|a\|})$. Now

$$\begin{aligned}
\Gamma_a(b_1) - \Gamma_a(b_2) &= b_1 - b_2 + \big(E(b_1ab_1a) - E(b_2ab_2a) \\
&\quad + E(b_1ab_1ab_1a) - E(b_2ab_2ab_2a) + \cdots)\big)E(a)^{-1} \\
&= b_1 - b_2 + \big(E((b_1-b_2)ab_1a) + E(b_2a(b_1-b_2)a) \\
&\quad + E(b_1-b_2)ab_1ab_1a) + E(b_2a(b_1-b_2)ab_2a) \\
&\quad + E(b_2ab_2a(b_1-b_2)a) + \cdots\big)E(a)^{-1}.
\end{aligned}$$

Defining $c = \|a\| \max\{\|b_1\|, \|b_2\|\}$ we estimate

$$\begin{aligned}
&\|\Psi_a(b_1) - \Psi_a(b_2) - (b_1 - b_2)\| \\
&\qquad \leq 2\|a\| \max\{\|b_1\|, \|b_2\|\} \|b_1 - b_2\| \|a\| \left\|E(a)^{-1}\right\| \\
&\qquad + 3\|a\|^2 \max\{\|b_1\|, \|b_2\|\}^2 \|b_1 - b_2\| \|a\| \left\|E(a)^{-1}\right\| \\
&\qquad\qquad + \cdots \\
&\qquad \leq \|b_1 - b_2\| \|a\| \left\|E(a)^{-1}\right\| \left(\sum_{n=0}^{\infty}(n+1)c^n - 1\right) \\
&\qquad = \left(\frac{2-c}{(1-c)^2}\right) c\|a\| \left\|E(a)^{-1}\right\| \|b_1 - b_2\| \\
&\qquad\qquad < \frac{\alpha(2-\alpha)}{(1-\alpha)^2}\|b_1 - b_2\|.
\end{aligned}$$

We infer that when $\alpha < \frac{1}{4}$ then $\frac{\alpha(2-\alpha)}{(1-\alpha)^2} < 1$ and the lemma follows. $\square$

Lemma 4.2. *Let $a \in \mathcal{A}$ be fixed. Let $\Psi_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ be the function defined by (4.1). Then Ψ_a is Fréchet differentiable, and the differential of Ψ_a is*

$$D\Psi_a(b) : h \mapsto E((1-ba)^{-1}ha(1-ba)^{-1})$$

for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ and $h \in \mathcal{B}$. Furthermore the differential, $D\Psi_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$, is continuous, so that Ψ_a is differentiable of class C^1. If $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ then

$$\|D\Psi_a(b) - D\Psi_a(0)\| < \frac{\|b\| \|a\|^2 (2 - \|b\| \|a\|)}{(1 - \|b\| \|a\|)^2}. \tag{4.2}$$

Proof. Recall that for $b_1, b_2 \in \mathcal{B}(0, \frac{1}{\|a\|})$ we have

$$(1-b_1a)^{-1} - (1-b_2a)^{-1} = (1-b_1a)^{-1}(b_1-b_2)a(1-b_2a)^{-1},$$

so

$$\begin{aligned} &\left\|(1-b_1a)^{-1} - (1-b_2a)^{-1}\right\| \\ &\qquad\qquad \le \frac{1}{1-\|b_1\|\|a\|}\frac{1}{1-\|b_2\|\|a\|}\|a\|\|b_1-b_2\| \to 0 \end{aligned} \tag{4.3}$$

for $b_1 \to b_2$ in norm. Also

$$\begin{aligned} \Psi_a(b_1) - \Psi_a(b_2) &= \sum_{n=1}^{\infty} E((b_1a)^n) - \sum_{n=1}^{\infty} E((b_2a)^n) \\ &= E\left((1-b_1a)^{-1} - (1-b_2a)^{-1}\right) \\ &= E\left((1-b_1a)^{-1}(b_1-b_2)a(1-b_2a)^{-1}\right). \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) we get

$$\begin{aligned} \Psi_a(b+h) - \Psi_a(b) &= E\left((1-(b+h)a)^{-1}ha(1-ba)^{-1}\right) \\ &= E\left((1-ba)^{-1}ha(1-ba)^{-1}\right) \\ &\quad + E\left(\left((1-(b+h)a)^{-1} - (1-ba)^{-1}\right)ha(1-ba)^{-1}\right) \\ &= E\left((1-ba)^{-1}ha(1-ba)^{-1}\right) + O(\|h\|^2) \end{aligned}$$

for all $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ and $h \in \mathcal{B}$ of small norm. This shows the first part of the lemma.

Let $b_1, b_2 \in \mathcal{B}(0, \frac{1}{\|a\|})$. Continuity of $D\Psi_a : b \mapsto D\Psi_a(b)$ follows from

$$\begin{aligned}
(4.5) \quad & \|D\Psi_a(b_1) - D\Psi_a(b_2)\| \\
&= \sup_{\|h\|\le 1} \|D\Psi_a(b_1)(h) - D\Psi_a(b_2)(h)\| \\
&= \sup_{\|h\|\le 1} \left\|E\left((1-b_1a)^{-1}ha(1-b_1a)^{-1}\right) - E\left((1-b_2a)^{-1}ha(1-b_2a)^{-1}\right)\right\| \\
&\le \sup_{\|h\|\le 1} \left\|(1-b_1a)^{-1}ha(1-b_1a)^{-1} - (1-b_1a)^{-1}ha(1-b_2a)^{-1}\right\| \\
&+ \le \sup_{\|h\|\le 1} \left\|(1-b_1a)^{-1}ha(1-b_2a)^{-1} - (1-b_2a)^{-1}ha(1-b_2a)^{-1}\right\| \\
&\le \left\|(1-b_1a)^{-1} - (1-b_2a)^{-1}\right\| \|a\| \left(\left\|(1-b_1a)^{-1}\right\| + \left\|(1-b_2a)^{-1}\right\|\right) \\
&\to 0
\end{aligned}$$

for $b_1 \to b_2$ in norm. Specifically letting $b_2 = 0$ and $b_1 = b$ in (4.5) we have

$$\begin{aligned}
\|D\Psi_a(b) - D\Psi_a(0)\| &\le \left\|(1-ba)^{-1} - 1\right\| \|a\| \left(\left\|(1-ba)^{-1}\right\| + 1\right) \\
&\le \|a\| \frac{\|b\|\,\|a\|}{1-\|b\|\,\|a\|} \left(\frac{1}{1-\|b\|\,\|a\|} + 1\right) \\
&= \frac{\|b\|\,\|a\|^2 (2 - \|b\|\,\|a\|)}{(1-\|b\|\,\|a\|)^2}.
\end{aligned}$$

□

Proposition 4.3. *Let $\mathcal{A}$ be a unital Banach algebra, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital Banach sub-algebra and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation. Let $a \in \mathcal{A}$ be a fixed element and assume that $E(a) \in \mathcal{B}_{\mathrm{inv}}$. The function $\Psi_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ defined by $\Psi_a(b) = E((1-ba)^{-1}) - 1$ for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ is a bijection of the neighborhood $\mathcal{B}(0, \frac{1}{4\|a\|^2\|E(a)^{-1}\|})$ onto a neighborhood of 0 which contains $\mathcal{B}(0, \frac{1}{11\|a\|^2\|E(a)^{-1}\|^2})$. Furthermore*

$$\Psi_a^{<-1>}\left(\mathcal{B}(0, \tfrac{1}{11\|a\|^2\|E(a)^{-1}\|^2})\right) \subseteq \mathcal{B}(0, \tfrac{2}{11\|a\|^2\|E(a)^{-1}\|}) \tag{4.6}$$

$$\Psi_a^{<-1>}\left(\mathcal{B}(0, \tfrac{1}{11\|a\|^2\|E(a)^{-1}\|^2})_{\mathrm{inv}}\right) \subseteq \mathcal{B}(0, \tfrac{2}{11\|a\|^2\|E(a)^{-1}\|})_{\mathrm{inv}} \tag{4.7}$$

Proof. The proof is very similar to the proof of proposition 2.3 and is again an inspection of some of the proof of the Inverse Function theorem. We only give the changes.

Define $\Gamma_a : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ by $\Gamma_a : b \mapsto \Psi_a(b)E(a)^{-1}$. Then we have $D\Gamma(0)(h) = h$. We now define $T : \mathcal{B}(0, \frac{1}{\|a\|}) \to \mathcal{B}$ by $T(b) = b - \Gamma_a(b)$

for $b \in \mathcal{B}(0, \frac{1}{\|a\|})$ and observe that $T(0) = 0$ and $DT(0) = 0$. Thus by use of (4.2) we have

$$\begin{aligned}\|DT(0)\| &= \|D\Gamma_a(b) - D\Gamma_a(0)\| \\ &\leq \left\|E(a)^{-1}\right\| \|D\Psi_a(b) - D\Psi_a(0)\| \\ &\leq \left\|E(a)^{-1}\right\| \|a\|^2 \|b\| \frac{2 - \|b\| \|a\|}{(1 - \|b\| \|a\|)^2} < \frac{40}{81} < \frac{1}{2}\end{aligned}$$

for $b \in \mathcal{B}(0, \frac{2}{11\|a\|^2\|E(a)^{-1}\|})$.

We can now proceed exactly as in the proof of proposition 2.3 to show that Γ_a maps $\mathcal{B}(0, \frac{2}{11\|a\|^2\|E(a)^{-1}\|})$ injectively onto a neighborhood of 0 containing $\mathcal{B}(0, \frac{1}{11\|a\|^2\|E(a)^{-1}\|})$. Thus if $b_0 \in \mathcal{B}(0, \frac{1}{11\|a\|^2\|E(a)^{-1}\|^2})$ it is obvious that $b_0 E(a)^{-1} \in \mathcal{B}(0, \frac{1}{11\|a\|^2\|E(a)^{-1}\|})$ so there exists $b \in \mathcal{B}(0, \frac{2}{11\|a\|^2\|E(a)^{-1}\|})$ such that $\Gamma_a(b) = b_0 E(a)^{-1}$. But then $\Psi_a(b) = \Gamma_a(b)E(a) = b_0$, so Ψ_a maps $\mathcal{B}(0, \frac{2}{11\|a\|^2\|E(a)^{-1}\|})$ injectively onto a neighborhood of 0 containing $\mathcal{B}(0, \frac{1}{11\|a\|^2\|E(a)^{-1}\|^2})$.

To see (4.7) observe that for $b \in \mathcal{B}(0, \frac{2}{11\|a\|^2\|E(a)^{-1}\|})$ we have

$$\begin{aligned}\left\|E(a)^{-1}\sum_{n=1}^{\infty} E(a(ba)^n)\right\| &\leq \left\|E(a)^{-1}\right\| \|a\|^2 \|b\| \sum_{n=0}^{\infty}(\|b\| \|a\|)^n \\ &< \frac{\frac{2}{11}}{1 - \frac{2}{11}} < 1,\end{aligned}$$

so (4.7) now follows from

$$\Psi_a(b) = \sum_{n=1}^{\infty} E((ba)^n) = bE(a)\left(1 + E(a)^{-1}\sum_{n=1}^{\infty} E(a(ba)^n)\right). \tag{4.8}$$

□

We can now define the amalgamated S-transform

Definition 4.4. Let $\mathcal{A}$ be a unital Banach algebra, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital commutative Banach sub-algebra and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation. Let $a \in \mathcal{A}$ be a fixed non-zero element such that $E(a) \in \mathcal{B}_{\text{inv}}$. Define the amalgamated S-transform of a by

$$S_a(b) := b^{-1}(1 + b)\Psi_a^{<-1>}(b) \tag{4.9}$$

for $b \in \mathcal{B}(0, \frac{1}{11\|a\|^2\|E(a)^{-1}\|^2})_{\text{inv}}$.

We have the following amalgamated version of (1.2).

Theorem 4.5. *Let $\mathcal{A}$ be a unital Banach algebra, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital commutative Banach sub-algebra and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation. Let $a_1, a_2 \in \mathcal{A}$ be $\mathcal{B}$-free fixed non-zero elements such that $E(a_1), E(a_2) \in \mathcal{B}_{\rm inv}$. Let*
$b \in \mathcal{B}(0, \min(\frac{1}{11\|a_1\|^2\|E(a_1)^{-1}\|^2}, \frac{1}{11\|a_2\|^2\|E(a_2)^{-1}\|^2}, \frac{1}{11\|a_1+a_2\|^2\|E(a_1+a_2)^{-1}\|^2}))_{\rm inv}$.
Then

$$S_{a_1a_2}(b) = S_{a_1}(b)S_{a_2}(b).$$

Proof. Let
$b \in \mathcal{B}(0, \min(\frac{1}{11\|a_1\|^2\|E(a_1)^{-1}\|^2}, \frac{1}{11\|a_2\|^2\|E(a_2)^{-1}\|^2}, \frac{1}{11\|a_1+a_2\|^2\|E(a_1+a_2)^{-1}\|^2}))_{\rm inv}$,
and let $b_1 \in \mathcal{B}(0, \frac{2}{11\|a_1\|^2\|E(a_1)^{-1}\|})_{\rm inv}$ and $b_2 \in \mathcal{B}(0, \frac{2}{11\|a_2\|^2\|E(a_2)^{-1}\|})_{\rm inv}$ be the uniquely determined elements such that

$$b = \Psi_{a_1}(b_1) = \Psi_{a_2}(b_2).$$

Note that

$$b + 1 = E((1 - b_1a_1)^{-1}) = E((1 - a_2b_2)^{-1}), \tag{4.10}$$

where the last equality follows since $\mathcal{B}$ is commutative. Define

$$\begin{aligned} A_1(b_1) &= (1 - b_1a_1)^{-1} - E((1 - b_1a_1)^{-1}), \\ A_2(b_2) &= (1 - a_2b_2)^{-1} - E((1 - a_2b_2)^{-1}). \end{aligned}$$

By (4.10) we have

$$\begin{aligned} (1 - b_1a_1)A_1(b_1) &= 1 - (1 - b_1a_1)(1 + b) \\ A_2(b_2)(1 - a_2b_2) &= 1 - (1 + b)(1 - a_2b_2), \end{aligned}$$

and thus

$$\begin{aligned} &(1 - b_1a_1)b\left(1 - b^{-1}A_1(b_1)(1 + b)^{-1}A_2(b_2)\right)(1 - a_2b_2) \\ &\qquad = (1 - b_1a_1)b(1 - a_2b_2) - \\ &\qquad \big(1 - (1 - b_1a_1)(1 + b)\big)(1 + b)^{-1}\big(1 - (1 + b)(1 - a_2b_2)\big) \\ &\qquad = -(1 + b)^{-1} + (1 - b_1a_1) + (1 - a_2b_2) - (1 - b_1a_1)(1 - a_2b_2) \\ &\qquad = 1 - (1 + b)^{-1} - b_1a_1a_2b_2 = \frac{b}{1 + b}\left(1 - \frac{1 + b}{b}b_1a_1a_2b_2\right), \end{aligned}$$

where $\frac{b}{1-b}$ and $\frac{1+b}{b}$ denotes the elements $(1 + b)^{-1}b$ and $b^{-1}(1 + b)$ respectively. We claim that $\|b^{-1}A_1(b_1)(1 + b)^{-1}A_2(b_2)\| < 1$. Inverting

we have

$$\left(1-\frac{1+b}{b}b_1a_1a_2b_2\right)^{-1}\frac{1+b}{b}$$
$$=(1-a_2b_2)^{-1}\left(1-b^{-1}A_1(b_1)(1+b)^{-1}A_2(b_2)\right)^{-1}b^{-1}(1-b_1a_1)^{-1}$$
$$(1-a_2b_2)^{-1}\sum_{n=0}^{\infty}(b^{-1}A_1(b_1)(1+b)^{-1}A_2(b_2))^n b^{-1}(1-b_1a_1)^{-1}.$$

Since $A_1(b_1)$ and $A_2(b_2)$ are $\mathcal{B}$-free and $E(A_1(b_1)) = E(A_2(b_2)) = 0$, $\mathcal{B}$-freeness implies by (1.4) that

$$E((1-\frac{1+b}{b}b_1a_1a_2b_2)^{-1})\frac{1+b}{b}$$
$$=E((1-a_2b_2)^{-1})b^{-1}E((1-b_1a_1)^{-1})=\frac{(1+b)^2}{b},$$

so we conclude that

$$\Psi_{a_1a_2}(\frac{1+b}{b}b_1b_2)=b.$$

Hence

$$S_{a_1a_2}(b)=\frac{1+b}{b}\left(\frac{1+b}{b}b_1b_2\right)=\left(\frac{1+b}{b}b_1\right)\left(\frac{1+b}{b}b_2\right)=S_{a_1}(b)S_{a_2}(b).$$

To prove the claim assume for a moment that

$$\|a_1\|\left\|E(a_1)^{-1}\right\|\leq\|a_2\|\left\|E(a_2)^{-1}\right\| \tag{4.11}$$

and observe that

$$b^{-1}A_1(b_1)=\Psi_{a_1}(b_1)^{-1}\left(\sum_{n=1}^{\infty}(b_1a_1)^n-\Psi_{a_1}(b_1)\right)$$
$$=\left(b_1E(a_1)\left(1+E(a_1)^{-1}\sum_{k=1}^{\infty}E(a_1(b_1a_1)^k)\right)\right)^{-1}\sum_{n=1}^{\infty}(b_1a_1)^n-1$$
$$=\left(1+E(a_1)^{-1}\sum_{k=1}^{\infty}E(a_1(b_1a_1)^k)\right)^{-1}E(a_1)^{-1}a_1\sum_{n=0}^{\infty}(b_1a_1)^n-1,$$

so

$$\left\|b^{-1}A_1(b_1)\right\|<\left(\frac{1}{1-\frac{2}{9}}\right)\|a_1\|\left\|E(a_1)^{-1}\right\|\left(\frac{1}{1-\frac{2}{11}}\right)+1 \tag{4.12}$$
$$=\frac{11}{7}\|a_1\|\left\|E(a_1)^{-1}\right\|+1\leq\frac{11}{7}\|a_2\|\left\|E(a_2)^{-1}\right\|+1.$$

Also

$$\begin{aligned}(1+b)^{-1}A_2(b_2) &= (1+b)^{-1}(1-a_2b_2)^{-1}-1\\ &= \sum_{n=0}^{\infty}(-b)^n\sum_{k=0}^{\infty}(a_2b_2)^k-1\\ &= \sum_{n=1}^{\infty}(-b)^n+\sum_{k=1}^{\infty}(a_2b_2)^k+\sum_{n=1}^{\infty}(-b)^n\sum_{k=1}^{\infty}(a_2b_2)^k,\end{aligned}$$

so

$$\begin{aligned}&\left\|(1+b)^{-1}A_2(b_2)\right\|\\ &\quad<\left(\frac{1}{10}+\frac{2}{9}+\frac{1}{10}\frac{2}{9}\right)\frac{1}{\|a_2\|\,\|E(a_2)^{-1}\|}=\frac{32}{90}\frac{1}{\|a_2\|\,\|E(a_2)^{-1}\|}.\end{aligned}\tag{4.13}$$

The claim now follows by combining (4.12) and (4.13)

$$\begin{aligned}&\left\|b^{-1}A_1(b_1)(1+b)^{-1}A_2(b_2)\right\|\\ &\qquad\leq\left\|b^{-1}A_1(b_1)\right\|\left\|(1+b)^{-1}A_2(b_2)\right\|\\ &\qquad<\left(\frac{11}{7}\|a_2\|\left\|E(a_2)^{-1}\right\|+1\right)\frac{32}{90}\frac{1}{\|a_2\|\,\|E(a_2)^{-1}\|}\\ &\qquad<\frac{11}{7}\frac{32}{90}+\frac{32}{90}<1.\end{aligned}$$

Finally if we have $\|a_1\|\,\|E(a_1)^{-1}\|\leq\|a_2\|\,\|E(a_2)^{-1}\|$ we just do similar calculations on $A_1(b_1)(1+b)^{-1}A_2(b_2)b^{-1}$ instead. □

5. Examples

The amalgamated R- and S-transform is related as follows.

Example 5.1. Let $\mathcal{A}$ be a unital Banach algebra, and let $1\in\mathcal{B}\subset\mathcal{A}$ be a unital commutative Banach sub-algebra, and $E:\mathcal{A}\to\mathcal{B}$ a conditional expectation. Let $a\in\mathcal{A}$ be a fixed non-zero element such that $E(a)\in\mathcal{B}_{\text{inv}}$. As in the scalar case [HL, NS2] we have the following relation beween the amalgamated R- and S-transform

$$bS_a(b)=[bR_a(b)]^{<-1>}.\tag{5.1}$$

To see (5.1) note that for $b\in\mathcal{B}_{\text{inv}}$ of small norm we have

$$\begin{aligned}g_a(b)R_a(g_a(b)) &= g_a(b)(b^{-1}-g_a(b)^{-1})\\ &= E((1-ba)^{-1})bb^{-1}-1\\ &= \sum_{n=1}^{\infty}E((ba)^n)=\Psi_a(b).\end{aligned}$$

and

$$\begin{aligned}\Psi_a(b)S_a(\Psi_a(b)) &= \Psi_a(b)\frac{1+\Psi_a(b)}{\Psi_a(b)}b \\ &= \left(1+\sum_{n=1}^{\infty}E((ba)^n)\right)b \\ &= g_a(b).\end{aligned}$$

So (5.1) now follows easily because

$$g_a(b)R_a(g_a(b))S_a\big(g_a(b)R_a(g_a(b))\big) = \Psi_a(b)S_a(\Psi(b)) = g_a(b)$$

and

$$\Psi_a(b)S_a(\Psi_a(b))R_a\big(\Psi_a(b)S_a(\Psi_a(b))\big) = g_a(b)R_a(g_a(b)) = \Psi_a(b).$$

□

We have the following dilation formula.

Example 5.2 (Dilations). Let $\mathcal{A}$ be a unital Banach algebra, and let $1 \in \mathcal{B} \subset \mathcal{A}$ be a unital commutative Banach sub-algebra, and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation. Let $a \in \mathcal{A}$ be a fixed non-zero element such that $E(a) \in \mathcal{B}_{\text{inv}}$. Assume that $z \in \mathcal{B}_{\text{inv}}$.

Then for $b \in \mathcal{B}(0, \frac{1}{\|za\|})$ and $bz \in \mathcal{B}(0, \frac{1}{\|a\|})$ we have

$$\Psi_{za}(b) = \sum_{n=1}^{\infty}E((b(za))^n) = \sum_{n-1}^{\infty}E(((bz)a)^n) = \Psi_a(bz),$$

and

$$S_z(\Psi_z(b)) = \left(\sum_{n=1}^{\infty}(bz)^n\right)^{-1}\left(\sum_{n=0}^{\infty}(bz)^n\right)bzz^{-1} = z^{-1},$$

for $\|b\|$ sufficiently small. Thus

$$\begin{aligned}S_{za}(\Psi_{za}(b)) &= \Psi_{za}(b)^{-1}(1+\Psi_{za}(b))b \\ &= \Psi_{za}(b)^{-1}(1+\Psi_{za}(b))(bz)z^{-1} \\ &= S_a(\Psi_a(bz))S_z(\Psi_a(bz)) \\ &= S_a(\Psi_{za}(b))S_z(\Psi_{za}(b))\end{aligned}$$

so

$$S_{za}(b) = S_a(b)S_z(b)$$

for b invertible and $\|b\|$ sufficiently small.

□

Note that in the above example we did *not* use the commutativeness of $\mathcal{B}$, so the example above shows that if we define the amalgamated S-transform by (4.9) for non-commutative $\mathcal{B}$ we would actually have to look for a product formula of the form:

$$S_{a_1 a_2}(b) = S_{a_2}(b) S_{a_1}(b) \tag{5.2}$$

for a_1 $*$-free from a_2 and $\|b\|$ sufficiently small. This suggests that it is actually more natural to consider the inverse of the S-transform than the S-transform.

Unfortunately we do not know whether (5.2) is true or not when $\mathcal{B}$ is non-commutative, but our guess is that it is not true in general.

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